MODELS OF NEUTRINO MASSES AND MIXINGS: A PROGRESS REPORT

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ABSTRACT

We present some recent developments on model building for neutrino masses and mixings. In particular, we review tri-bimaximal neutrino mixing derived from discrete groups, notably A4. We discuss the problems encountered with extending the symmetry to the quark sector and with Grand Unification.

1. Introduction: "Normal" versus "Exceptional" Models

After KamLAND, SNO and WMAP not too much hierarchy in neutrino masses is indicated by experiments:

$$r = \Delta m^2_{sol}/\Delta m^2_{atm} \sim 1/30. \tag{1}$$

Precisely at 2σ: $0.025 \lesssim r \lesssim 0.049$ [1]. Thus, for a hierarchical spectrum, $m_2/m_3 \sim \sqrt{r} \sim 0.2$, which is comparable to the Cabibbo angle $\lambda_C \sim 0.22$ or $\sqrt{m_\mu/m_\tau} \sim 0.24$. This suggests that the same hierarchy parameter (raised to powers with o(1) exponents) may apply for quark, charged lepton and neutrino mass matrices. This in turn indicates that, in absence of some special dynamical reason, we do not expect quantities like θ_{13} or the deviation of θ_{23} from its maximal value to be too small. Indeed it would be very important to know how small the mixing angle θ_{13} is and how close to maximal θ_{23} is. Actually one can make a distinction between "normal" and "exceptional" models. For normal models θ_{23} is not too close to maximal and θ_{13} is not too small, typically a small power of the self-suggesting order parameter $\sqrt{r}$, with $r = \Delta m^2_{sol}/\Delta m^2_{atm} \sim 1/30$. Exceptional models are those where some symmetry or dynamical feature assures in a natural way the near vanishing of θ_{13} and/or of $\theta_{23} - \pi/4$. Normal models are conceptually more economical and much simpler to construct. Typical categories of normal models are (we refer to the review in ref.[2] for a detailed discussion of the relevant models and a more complete list of references):

a) Anarchy. These are models with approximately degenerate mass spectrum and no ordering principle or approximate symmetry assumed in the neutrino mass sector [3]. The small value of r is accidental, due to random fluctuations of matrix elements in the Dirac and Majorana neutrino mass matrices. Starting from a random input for each matrix element, the see-saw formula, being a product of 3 matrices, generates a broad distribution of r values. All mixing angles are generically large: so in this case one does not expect θ_{23} to be maximal and θ_{13} should probably be found near its upper bound.

b) Semianarchy. We have seen that anarchy is the absence of structure in the neutrino sector. Here we consider an attenuation of anarchy where the absence of structure is limited to the 23 neutrino sector. The typical structure is in this case [4]:

$$m_\nu \approx m \left(\begin{array}{ccc} \delta & \epsilon & \epsilon \\ \epsilon & 1 & 1 \\ \epsilon & 1 & 1 \end{array} \right) \quad , \tag{2}$$

where δ and ϵ are small and by 1 we mean entries of $o(1)$ and also the 23 determinant is of $o(1)$. This texture can be realized, for example, without see-saw from a suitable set of $U(1)_F$ charges for (l_1, l_2, l_3), eg $(a, 0, 0)$ appearing in the dim. 5 operator $\lambda l^T l HH/M$. Clearly, in general we would expect two mass eigenvalues of order 1, in units of m, and one small, of order δ or ϵ^2. This typical pattern would not fit the observed solar and atmospheric observed frequencies. However, given that $\sqrt{r}$ is not too small, we can assume that its small value is generated accidentally, as for anarchy. We see that, if by chance the second eigenvalue $\eta \sim \sqrt{r} \sim \delta + \epsilon^2$, we can then obtain the correct value of r together with large but in general non maximal θ_{23} and θ_{12} and small $\theta_{13} \sim \epsilon$. The guaranteed smallness of θ_{13} is the main advantage over anarchy, and the relation with $\sqrt{r}$ normally keeps θ_{13} not too small. For example, $\delta \sim \epsilon^2$ in typical $U(1)_F$ models that provide a very economical but effective realization of this scheme .

c) Inverse hierarchy. One obtains inverted hierarchy, for example, in the limit of exact $L_e - L_\mu - L_\tau$ symmetry [5]. In this limit $r = 0$ and θ_{12} is maximal while θ_{23} is generically large. [2]. Simple forms of symmetry breaking cannot sufficiently displace θ_{12} from the maximal value because typically $\tan^2\theta_{12} \sim 1+o(r)$. Viable normal models can be obtained by arranging large contributions to θ_{23} and θ_{12} from the charged lepton mass diagonalization. But then, it turns out that, in order to obtain the measured value of θ_{12}, the size of θ_{13} must be close to its present upper bound [6]. If indeed the shift from maximal θ_{12} is due to the charged lepton diagonalization, this could offer a possible track to explain the empirical relation $\theta_{12} + \theta_C = \pi/4$ [7] (with present data $\theta_{12} + \theta_C = (47.0 + 1.7 - 1.6)^0$). While it would not be difficult in this case to

arrange that the shift from maximal is of the order of θ_C, it is not clear how to guarantee that it is precisely equal to θ_C [8]. Besides the effect of the charged lepton diagonalization, in a see-saw context, one can assume a strong additional breaking of $L_e - L_\mu - L_\tau$ from soft terms in the M_{RR} Majorana mass matrix [9]. Since ν_R's are gauge singlets and thus essentially uncoupled, a large breaking in M_{RR} does not feedback in other sectors of the lagrangian. In this way one can obtain realistic values for θ_{12} and for all other masses and mixings, in particular also with a small θ_{13}.

d) Normal hierarchy. Particularly interesting are models with 23 determinant suppressed by see-saw [2]: in the 23 sector one needs relatively large mass splittings to fit the small value of r but nearly maximal mixing. This can be obtained if the 23 sub-determinant is suppressed by some dynamical trick. Typical examples are lopsided models with large off diagonal term in the Dirac matrices of charged leptons and/or neutrinos (in minimal SU(5) the d-quark and charged lepton mass matrices are one the transposed of the other, so that large left-handed mixings for charged leptons correspond to large unobservable right-handed mixings for d-quarks). Another class of typical examples is the dominance in the see-saw formula of a small eigenvalue in M_{RR}, the right-handed Majorana neutrino mass matrix. When the 23 determinant suppression is implemented in a 3x3 context, normally θ_{13} is not protected from contributions that vanish with the 23 determinant, hence with r.

The fact that some neutrino mixing angles are large and even nearly maximal, while surprising at the start, was soon realised to be well compatible with a unified picture of quark and lepton masses within GUTs. The symmetry group at M_{GUT} could be either (SUSY) SU(5) or SO(10) or a larger group. For example, normal models based on anarchy, semianarchy, inverted hierarchy or normal hierarchy can all be naturally implemented by simple assignments of $\mathrm{U(1)_F}$ horizontal charges in a semiquantitative unified description of all quark and lepton masses in SUSY SU(5)$\times$ $\mathrm{U(1)_F}$. Actually, in this context, if one adopts a statistical criterium, hierarchical models appear to be preferred over anarchy and among them normal hierarchy with see-saw ends up as being the most likely [10].

In conclusion we expect that experiment will eventually find that θ_{13} is not too small and that θ_{23} is sizably not maximal. But if, on the contrary, either θ_{13} is found from experiment to be very small or θ_{23} to be very close to maximal or both, then theory will need to cope with this fact. Normal models have been extensively discussed in the literature [2], so we concentrate here on a particularly interesting class

of exceptional models.

2. Tri-bimaximal Mixing

Here we want to discuss particular exceptional models where both θ_{13} and $\theta_{23}-\pi/4$ exactly vanish (more precisely, they vanish in a suitable limit, with correction terms that can be made negligibly small) and, in addition, $s_{12} \sim 1/\sqrt{3}$, a value which is in very good agreement with present data. This is the so-called tri-bimaximal or Harrison-Perkins-Scott mixing pattern (HPS) [11], with the entries in the second column all equal to $1/\sqrt{3}$ in absolute value. Here we adopt the following phase convention:

$$U_{HPS} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} \quad . \tag{3}$$

In the HPS scheme $\tan^2\theta_{12} = 0.5$, to be compared with the latest experimental determination [1]: $\tan^2\theta_{12} = 0.46^{+0.06}_{-0.05}$ (at 1σ). Thus the HPS mixing matrix is a good representation of the present data within one σ. The challenge is to find natural and appealing schemes that lead to this matrix with good accuracy. Clearly, in a natural realization of this model, a very constraining and predictive dynamics must be underlying. It is interesting to explore particular structures giving rise to this very special set of models in a natural way. In this case we have a maximum of "order" implying special values for all mixing angles. Interesting ideas on how to obtain the HPS mixing matrix have been discussed in refs. [11,12,13]. Some attractive models are based on the discrete symmetry A4, which appears as particularly suitable for the purpose, and were presented in ref. [14,15,16,17,18,19].

The HPS mixing matrix suggests that mixing angles are independent of mass ratios (while for quark mixings relations like $\lambda_C^2 \sim m_d/m_s$ are typical). In fact in the basis where charged lepton masses are diagonal, the effective neutrino mass matrix in the HPS case is given by $m_\nu = U_{HPS}\mathrm{diag}(m_1, m_2, m_3)U^T_{HPS}$:

$$m_\nu = \left[\frac{m_3}{2}M_3 + \frac{m_2}{3}M_2 + \frac{m_1}{6}M_1\right] \quad . \tag{4}$$

where:

$$M_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 4 & -2 & -2 \\ -2 & 1 & 1 \\ -2 & 1 & 1 \end{pmatrix}. \tag{5}$$

The eigenvalues of m_ν are m_1, m_2, m_3 with eigenvectors $(-2,1,1)/\sqrt{6}$, $(1,1,1)/\sqrt{3}$ and $(0,1,-1)/\sqrt{2}$, respectively. In general, disregarding possible Majorana phases,

there are six parameters in a real symmetric matrix like m_ν: here only three are left after the values of the three mixing angles have been fixed à la HPS. For a hierarchical spectrum $m_3 >> m_2 >> m_1$, $m_3^2 \sim \Delta m^2_{atm}$, $m_2^2/m_3^2 \sim \Delta m^2_{sol}/\Delta m^2_{atm}$ and m_1 could be negligible. But also degenerate masses and inverse hierarchy can be reproduced: for example, by taking $m_3 = -m_2 = m_1$ we have a degenerate model, while for $m_1 = -m_2$ and $m_3 = 0$ an inverse hierarchy case is realized (stability under renormalization group running strongly prefers opposite signs for the first and the second eigenvalue which are related to solar oscillations and have the smallest mass squared splitting).

It is interesting to recall that the most general mass matrix, in the basis where charged leptons are diagonal, that corresponds to $\theta_{13} = 0$ and θ_{23} maximal is of the form [21]:

$$m = \begin{pmatrix} x & y & y \\ y & z & w \\ y & w & z \end{pmatrix}, \tag{6}$$

Note that this matrix is symmetric under 2-3 or $\mu - \tau$ exchange [22]. For $\theta_{13} = 0$ there is no CP violation, so that, disregarding Majorana phases, we can restrict our consideration to real parameters. There are four of them in eq.(6) which correspond to three mass eigenvalues and one remaining mixing angle, θ_{12}. In particular, θ_{12} is given by:

$$\sin^2 2\theta_{12} = \frac{8y^2}{(x-w-z)^2 + 8y^2} \tag{7}$$

In the HPS case $\sin^2 2\theta_{12} = 8/9$ is also fixed and an additional parameter can be eliminated, leading to:

$$m = \begin{pmatrix} x & y & y \\ y & x+v & y-v \\ y & y-v & x+v \end{pmatrix}, \tag{8}$$

It is easy to see that the HPS mass matrix in eqs.(4-5) is indeed of the form in eq.(8).

In the next sections we will present models of tri-bimaximal mixing based on the A4 group. We first introduce A4 and its representations and then we show that this group is particularly suited to the problem.

3. The A4 Group

A4 is the group of the even permutations of 4 objects. It has 4!/2=12 elements. Geometrically, it can be seen as the invariance group of a tethraedron (the odd permutations, for example the exchange of two vertices, cannot be obtained by moving a rigid solid). Let us denote a generic permutation $(1, 2, 3, 4) \to (n_1, n_2, n_3, n_4)$ simply by $(n_1 n_2 n_3 n_4)$. $A4$ can be generated by two basic permutations S and T given by $S = (4321)$ and $T = (2314)$. One checks immediately that:

$$S^2 = T^3 = (ST)^3 = 1 \tag{9}$$

Table 1: Characters of A4

Class	χ^1	$\chi^{1'}$	$\chi^{1"}$	χ^3
C_1	1	1	1	3
C_2	1	ω	ω^2	0
C_3	1	ω^2	ω	0
C_4	1	1	1	-1

This is called a "presentation" of the group. The 12 even permutations belong to 4 equivalence classes (h and k belong to the same class if there is a g in the group such that $ghg^{-1} = k$) and are generated from S and T as follows:

$$\begin{aligned}
C1 \quad &: I = (1234) \\
C2 \quad &: T = (2314), ST = (4132), TS = (3241), STS = (1423) \\
C3 \quad &: T^2 = (3124), ST^2 = (4213), T^2S = (2431), TST = (1342) \\
C4 \quad &: S = (4321), T^2ST = (3412), TST^2 = (2143)
\end{aligned} \tag{10}$$

Note that, except for the identity I which always forms an equivalence class in itself, the other classes are according to the powers of T (in C4 S could as well be seen as ST^3).

In a finite group the squared dimensions of the inequivalent irreducible representations add up to N, the number of transformations in the group ($N = 12$ in $A4$). $A4$ has four inequivalent representations: three of dimension one, 1, $1'$ and 1" and one of dimension 3. It is immediate to see that the one-dimensional unitary representations are obtained by:

$$\begin{aligned}
1 \quad & S = 1 \quad T = 1 \\
1' \quad & S = 1 \quad T = e^{i2\pi/3} \equiv \omega \\
1'' \quad & S = 1 \quad T = e^{i4\pi/3} \equiv \omega^2
\end{aligned} \tag{11}$$

Note that $\omega = -1/2 + \sqrt{3}/2$ is the cubic root of 1 and satisfies $\omega^2 = \omega^*$, $1+\omega+\omega^2 = 0$.

The three-dimensional unitary representation, in a basis where the element S is diagonal, is built up from:

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \; T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{12}$$

The characters of a group χ_g^R are defined, for each element g, as the trace of the matrix that maps the element in a given representation R. It is easy to see that equivalent representations have the same characters and that characters have the same

value for all elements in an equivalence class. Characters satisfy $\sum_g \chi_g^R \chi_g^{S*} = N\delta^{RS}$. Also, for each element h, the character of h in a direct product of representations is the product of the characters: $\chi_h^{R\otimes S} = \chi_h^R \chi_h^S$ and also is equal to the sum of the characters in each representation that appears in the decomposition of $R \otimes S$. The character table of A4 is given in Table II [14]. From this Table one derives that indeed there are no more inequivalent irreducible representations other than 1, $1'$, 1" and 3. Also, the multiplication rules are clear: the product of two 3 gives $3 \times 3 = 1 + 1' + 1'' + 3 + 3$ and $1' \times 1' = 1''$, $1' \times 1'' = 1$, $1'' \times 1'' = 1'$ etc. If $3 \sim (a_1, a_2, a_3)$ is a triplet transforming by the matrices in eq.(12) we have that under S: $S(a_1, a_2, a_3)^t = (a_1, -a_2, -a_3)^t$ (here the upper index t indicates transposition) and under T: $T(a_1, a_2, a_3)^t = (a_2, a_3, a_1)^t$. Then, from two such triplets $3_a \sim (a_1, a_2, a_3)$, $3_b \sim (b_1, b_2, b_3)$ the irreducible representations obtained from their product are:

$$1 = a_1 b_1 + a_2 b_2 + a_3 b_3 \tag{13}$$

$$1' = a_1 b_1 + \omega^2 a_2 b_2 + \omega a_3 b_3 \tag{14}$$

$$1" = a_1 b_1 + \omega a_2 b_2 + \omega^2 a_3 b_3 \tag{15}$$

$$3 \sim (a_2 b_3, a_3 b_1, a_1 b_2) \tag{16}$$

$$3 \sim (a_3 b_2, a_1 b_3, a_2 b_1) \tag{17}$$

In fact, take for example the expression for $1" = a_1 b_1 + \omega a_2 b_2 + \omega^2 a_3 b_3$. Under S it is invariant and under T it goes into $a_2 b_2 + \omega a_3 b_3 + \omega^2 a_1 b_1 = \omega^2 [a_1 b_1 + \omega a_2 b_2 + \omega^2 a_3 b_3]$ which is exactly the transformation corresponding to 1".

In eq.(12) we have the representation 3 in a basis where S is diagonal. It is interesting to go to a basis where instead it is T which is diagonal. This is obtained through the unitary transformation:

$$T' = VTV^\dagger = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \tag{18}$$

$$S' = VSV^\dagger = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}. \tag{19}$$

where:

$$V = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}. \tag{20}$$

The matrix V is special in that it is a 3x3 unitary matrix with all entries of unit absolute value. It is interesting that this matrix was proposed long ago as a possible mixing matrix for neutrinos [23]. We shall see in the following that the matrix V appears in $A4$ models as the unitary transformation that diagonalizes the charged lepton mass matrix.

There is an interesting relation [18] between the A_4 model considered so far and the modular group. This relation could possibly be relevant to understand the origin of the A4 symmetry from a more fundamental layer of the theory. The modular group Γ is the group of linear fractional transformations acting on a complex variable z:

$$z \to \frac{az+b}{cz+d} \quad , \qquad ad-bc=1 \quad , \tag{21}$$

where a, b, c, d are integers. There are infinite elements in Γ, but all of them can be generated by the two transformations:

$$s: \quad z \to -\frac{1}{z} \quad , \qquad t: \quad z \to z+1 \quad , \tag{22}$$

The transformations s and t in (22) satisfy the relations

$$s^2 = (st)^3 = 1 \tag{23}$$

and, conversely, these relations provide an abstract characterization of the modular group. Since the relations (9) are a particular case of the more general constraint (23), it is clear that A4 is a very small subgroup of the modular group and that the A4 representations discussed above are also representations of the modular group. In string theory the transformations (22) operate in many different contexts. For instance the role of the complex variable z can be played by a field, whose VEV can be related to a physical quantity like a compactification radius or a coupling constant. In that case s in eq. (22) represents a duality transformation and t in eq. (22) represent the transformation associated to an "axionic" symmetry.

A different way to understand the dynamical origin of A_4 was recently presented in ref. [19] where it is shown that the A_4 symmetry can be simply obtained by orbifolding starting from a model in 6 dimensions (6D) (see also [20]). In this approach A_4 appears as the remnant of the reduction from 6D to 4D space-time symmetry induced by the special orbifolding adopted. There are 4D branes at the four fixed points of the orbifolding and the tetrahedral symmetry of A_4 connects these branes. The standard model fields have components on the fixed point branes while the scalar fields necessary for the A_4 breaking are in the bulk. Each brane field, either a triplet or a singlet, has components on all of the four fixed points (in particular all components are equal for a singlet) but the interactions are local, i.e. all vertices involve products of field components at the same space-time point. This approach suggests a deep relation between flavour symmetry in 4D and space-time symmetry in extra dimensions. However, the specific classification of the fields under A4 which is adopted in our model does not follow from the compactification and is separately assumed.

The orbifolding is defined as follows. We consider a quantum field theory in 6 dimensions, with two extra dimensions compactified on an orbifold T^2/Z_2. We

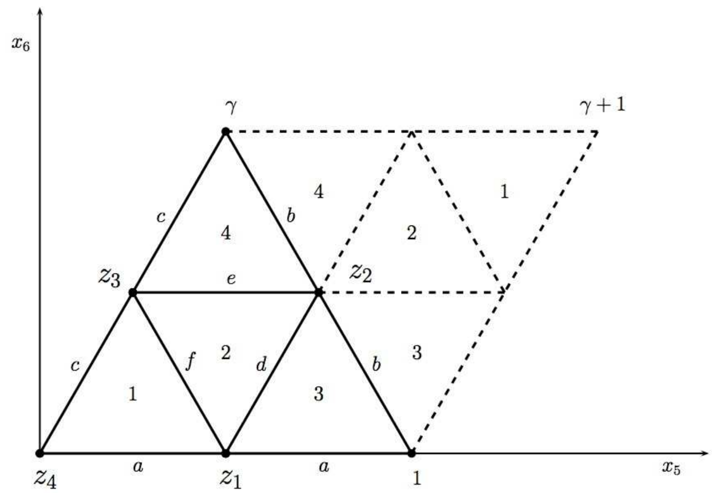

Figure 1: Orbifold T_2/Z_2. The regions with the same numbers are identified with each other. The four triangles bounded by solid lines form the fundamental region, where also the edges with the same letters are identified. The orbifold T_2/Z_2 is exactly a regular tetrahedron with 6 edges a, b, c, d, e, f and four vertices z_1, z_2, z_3, z_4, corresponding to the four fixed points of the orbifold.

denote by $z = x_5 + ix_6$ the complex coordinate describing the extra space. The torus T^2 is defined by identifying in the complex plane the points related by

$$\begin{aligned} &z \longrightarrow z+1 \\ &z \longrightarrow z+\gamma \qquad\qquad \gamma = e^{i\dfrac{\pi}{3}} \quad , \end{aligned} \tag{24}$$

where our length unit, $2\pi R$, has been set to 1 for the time being. The parity Z_2 is defined by

$$z \longrightarrow -z \tag{25}$$

and the orbifold T^2/Z_2 can be represented by the fundamental region given by the triangle with vertices $0, 1, \gamma$, see Fig. 1. The orbifold has four fixed points, $(z_1, z_2, z_3, z_4) = (1/2, (1+\gamma)/2, \gamma/2, 0)$. The fixed point z_4 is also represented by the vertices 1 and γ. In the orbifold, the segments labelled by a in Fig. 1, $(0, 1/2)$ and $(1, 1/2)$, are identified and similarly for those labelled by b, $(1, (1+\gamma)/2)$ and $(\gamma, (1+\gamma)/2)$, and those labelled by c, $(0, \gamma/2)$, $(\gamma, \gamma/2)$. Therefore the orbifold is a regular tetrahedron with vertices at the four fixed points.

The symmetry of the uncompactified 6D space time is broken by compactification. Here we assume that, before compactification, the space-time symmetry coincides with the product of 6D translations and 6D proper Lorentz transformations. The compactification breaks part of this symmetry. However, due to the special geometry

of our orbifold, a discrete subgroup of rotations and translations in the extra space is left unbroken. This group can be generated by two transformations:

$$\begin{array}{lll} \mathcal{S}: & z \to z + \frac{1}{2} & \\ \mathcal{T}: & z \to \omega z & \qquad \omega \equiv \gamma^2 \quad . \end{array} \tag{26}$$

Indeed $\mathcal{S}$ and $\mathcal{T}$ induce even permutations of the four fixed points:

$$\begin{array}{ll} \mathcal{S}: & (z_1, z_2, z_3, z_4) \to (z_4, z_3, z_2, z_1) \\ \mathcal{T}: & (z_1, z_2, z_3, z_4) \to (z_2, z_3, z_1, z_4) \end{array} \quad , \tag{27}$$

thus generating the group A_4. From the previous equations we immediately verify that $\mathcal{S}$ and $\mathcal{T}$ satisfy the characteristic relations obeyed by the generators of A_4: $\mathcal{S}^2 = \mathcal{T}^3 = (\mathcal{S}\mathcal{T})^3 = 1$. These relations are actually satisfied not only at the fixed points, but on the whole orbifold, as can be easily checked from the general definitions of $\mathcal{S}$ and $\mathcal{T}$ in eq. (26), with the help of the orbifold defining rules in eqs. (24) and (25).

4. Applying A4 to Lepton Masses and Mixings

A typical A4 model works as follows [17], [18]. One assigns leptons to the four inequivalent representations of A4: left-handed lepton doublets l transform as a triplet 3, while the right-handed charged leptons e^c, μ^c and τ^c transform as 1, $1'$ and $1''$, respectively. At this stage we do not introduce RH neutrinos, but later we will discuss a see-saw realization. The flavour symmetry is broken by two real triplets φ and φ' and by a real singlet ξ. These flavon fields are all gauge singlets. We also need one or two ordinary SM Higgs doublets $h_{u,d}$, which we take invariant under A4. The Yukawa interactions in the lepton sector read:

$$\begin{aligned} \mathcal{L}_Y &= y_e e^c (\varphi l) + y_\mu \mu^c (\varphi l)'' + y_\tau \tau^c (\varphi l)' \\ &+ x_a \xi (ll) + x_d (\varphi' ll) + h.c. + ... \end{aligned} \tag{28}$$

In our notation, (33) transforms as 1, $(33)'$ transforms as $1'$ and $(33)''$ transforms as $1''$. Also, to keep our notation compact, we use a two-component notation for the fermion fields and we set to 1 the Higgs fields $h_{u,d}$ and the cut-off scale Λ. For instance $y_e e^c (\varphi l)$ stands for $y_e e^c (\varphi l) h_d/\Lambda$, $x_a \xi (ll)$ stands for $x_a \xi (l h_u l h_u)/\Lambda^2$ and so on. The Lagrangian $\mathcal{L}_Y$ contains the lowest order operators in an expansion in powers of $1/\Lambda$. Dots stand for higher dimensional operators that will be discussed later. Some terms allowed by the flavour symmetry, such as the terms obtained by the exchange $\varphi' \leftrightarrow \varphi$, or the term (ll) are missing in $\mathcal{L}_Y$. Their absence is crucial and, in each version of A4 models, is motivated by additional symmetries. For example (ll), being of lower dimension with respect to $(\varphi' ll)$, would be the dominant component, proportional to the identity, of the neutrino mass matrix. In addition to that, the presence of the

singlet flavon ξ plays an important role in making the VEV directions of φ and φ' different.

For the model to work it is essential that the fields φ', φ and ξ develop a VEV along the directions:

$$\begin{aligned}\langle\varphi'\rangle &= (v', 0, 0)\\ \langle\varphi\rangle &= (v, v, v)\\ \langle\xi\rangle &= u \quad .\end{aligned} \tag{29}$$

A crucial part of all serious A4 models is the dynamical generation of this alignment in a natural way. If the alignment is realized, at the leading order of the $1/\Lambda$ expansion, the mass matrices m_l and m_ν for charged leptons and neutrinos are given by:

$$m_l = v_d \frac{v}{\Lambda} \left(\begin{array}{ccc} y_e & y_e & y_e \\ y_\mu & y_\mu\omega^2 & y_\mu\omega \\ y_\tau & y_\tau\omega & y_\tau\omega^2 \end{array} \right) \quad , \tag{30}$$

$$m_\nu = \frac{v_u^2}{\Lambda} \left(\begin{array}{ccc} a & 0 & 0 \\ 0 & a & d \\ 0 & d & a \end{array} \right) \quad , \tag{31}$$

where

$$a \equiv x_a \frac{u}{\Lambda} \quad , \qquad d \equiv x_d \frac{v'}{\Lambda} \quad . \tag{32}$$

Charged leptons are diagonalized by the matrix

$$l \rightarrow Vl = \frac{1}{\sqrt{3}} \left(\begin{array}{ccc} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{array} \right) l \quad , \tag{33}$$

This matrix was already introduced in eq.(20) as the unitary transformation between the S-diagonal to the T-diagonal 3x3 representation of $A4$. In fact, in this model, the S-diagonal basis is the Lagrangian basis and the T diagonal basis is that of diagonal charged leptons. The great virtue of $A4$ is to immediately produce the special unitary matrix V as the diagonalizing matrix of charged leptons and also to allow a singlet made up of three triplets, $(\phi' ll) = \phi'_1 l_2 l_3 + \phi'_2 l_3 l_1 + \phi'_3 l_1 l_2$ which leads, for the alignment in eq. (29), to the right neutrino mass matrix to finally obtain the HPS mixing matrix.

The charged fermion masses are given by:

$$m_e = \sqrt{3} y_e v_d \frac{v}{\Lambda} \quad , \qquad m_\mu = \sqrt{3} y_\mu v_d \frac{v}{\Lambda} \quad , \qquad m_\tau = \sqrt{3} y_\tau v_d \frac{v}{\Lambda} \quad . \tag{34}$$

We can easily obtain in a a natural way the observed hierarchy among m_e, m_μ and m_τ by introducing an additional $\mathrm{U}(1)_F$ flavour symmetry under which only the right-handed lepton sector is charged. We assign F-charges 0, 2 and $3 \div 4$ to τ^c, μ^c and

e^c, respectively. By assuming that a flavon θ, carrying a negative unit of F, acquires a VEV $\langle\theta\rangle/\Lambda \equiv \lambda < 1$, the Yukawa couplings become field dependent quantities $y_{e,\mu,\tau} = y_{e,\mu,\tau}(\theta)$ and we have

$$y_\tau \approx O(1) \quad , \qquad y_\mu \approx O(\lambda^2) \quad , \qquad y_e \approx O(\lambda^{3\div 4}) \quad . \tag{35}$$

In the flavour basis the neutrino mass matrix reads [notice that the change of basis induced by V, because of the Majorana nature of neutrinos, will in general change the relative phases of the eigenvalues of m_ν (compare eq.(31) with eq.(36))]:

$$m_\nu = \frac{v_u^2}{\Lambda}\left(\begin{array}{ccc} a+2d/3 & -d/3 & -d/3 \\ -d/3 & 2d/3 & a-d/3 \\ -d/3 & a-d/3 & 2d/3 \end{array}\right) \quad , \tag{36}$$

and is diagonalized by the transformation:

$$U^T m_\nu U = \frac{v_u^2}{\Lambda}\mathtt{diag}(a+d, a, -a+d) \quad , \tag{37}$$

with

$$U = \left(\begin{array}{ccc} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & +1/\sqrt{2} \end{array}\right) \quad . \tag{38}$$

The leading order predictions are $\tan^2\theta_{23} = 1$, $\tan^2\theta_{12} = 0.5$ and $\theta_{13} = 0$. The neutrino masses are $m_1 = a+d$, $m_2 = a$ and $m_3 = -a+d$, in units of v_u^2/Λ. We can express $|a|$, $|d|$ in terms of $r \equiv \Delta m^2_{sol}/\Delta m^2_{atm} \equiv (|m_2|^2 - |m_1|^2)/|m_3|^2 - |m_1|^2)$, $\Delta m^2_{atm} \equiv |m_3|^2 - |m_1|^2$ and $\cos\Delta$, Δ being the phase difference between the complex numbers a and d:

$$\begin{aligned} \sqrt{2}|a|\frac{v_u^2}{\Lambda} &= \frac{-\sqrt{\Delta m^2_{atm}}}{2\cos\Delta\sqrt{1-2r}} \\ \sqrt{2}|d|\frac{v_u^2}{\Lambda} &= \sqrt{1-2r}\sqrt{\Delta m^2_{atm}} \quad . \end{aligned} \tag{39}$$

To satisfy these relations a moderate tuning is needed in this model. Due to the absence of (ll) in eq. (28) which we will motivate in the next section, a and d are of the same order in $1/\Lambda$, see eq. (32). Therefore we expect that $|a|$ and $|d|$ are close to each other and, to satisfy eqs. (39), $\cos\Delta$ should be negative and of order one. We obtain:

$$\begin{aligned} |m_1|^2 &= \left[-r + \frac{1}{8\cos^2\Delta(1-2r)}\right]\Delta m^2_{atm} \\ |m_2|^2 &= \frac{1}{8\cos^2\Delta(1-2r)}\Delta m^2_{atm} \\ |m_3|^2 &= \left[1 - r + \frac{1}{8\cos^2\Delta(1-2r)}\right]\Delta m^2_{atm} \end{aligned} \tag{40}$$

If $\cos\Delta = -1$, we have a neutrino spectrum close to hierarchical:

$$|m_3| \approx 0.053 \ \ \text{eV} \quad , \qquad |m_1| \approx |m_2| \approx 0.017 \ \ \text{eV} \quad . \tag{41}$$

In this case the sum of neutrino masses is about 0.087 eV. If $\cos\Delta$ is accidentally small, the neutrino spectrum becomes degenerate. The value of $|m_{ee}|$, the parameter characterizing the violation of total lepton number in neutrinoless double beta decay, is given by:

$$|m_{ee}|^2 = \left[-\frac{1+4r}{9} + \frac{1}{8\cos^2\Delta(1-2r)} \right] \Delta m^2_{atm} \quad . \tag{42}$$

For $\cos\Delta = -1$ we get $|m_{ee}| \approx 0.005$ eV, at the upper edge of the range allowed for normal hierarchy, but unfortunately too small to be detected in a near future. Independently from the value of the unknown phase Δ we get the relation:

$$|m_3|^2 = |m_{ee}|^2 + \frac{10}{9}\Delta m^2_{atm}\left(1 - \frac{r}{2}\right) \quad , \tag{43}$$

which is a prediction of this model.

5. A4 model with an extra dimension

One of the problems we should solve in the quest for the correct alignment is that of keeping neutrino and charged lepton sectors separate, allowing φ and φ' to take different VEVs and also forbidding the exchange of one with the other in interaction terms. One possibility is that this separation is achieved by means of an extra spatial dimension, as discussed in ref. [17]. The space-time is assumed to be five-dimensional, the product of the four-dimensional Minkowski space-time times an interval going from $y = 0$ to $y = L$. At $y = 0$ and $y = L$ the space-time has two four-dimensional boundaries, called "branes". The idea is that matter SU(2) singlets such as e^c, μ^c, τ^c are localized at $y = 0$, while SU(2) doublets, such as l are localized at $y = L$ (see Fig.1). Neutrino masses arise from local operators at $y = L$. Charged lepton masses are produced by non-local effects involving both branes. The simplest possibility is to introduce a bulk fermion, depending on all space-time coordinates, that interacts with e^c, μ^c, τ^c at $y = 0$ and with l at $y = L$. The exchange of such a fermion can provide the desired non-local coupling between right-handed and left-handed ordinary fermions. Finally, assuming that φ and (φ', ξ) are localized respectively at $y = 0$ and $y = L$, one obtains a natural separation between the two sectors.

Such a separation also greatly simplifies the vacuum alignment problem. One can determine the minima of two scalar potentials V_0 and V_L, depending only, respectively, on φ and (φ', ξ). Indeed, it is shown that there are whole regions of the parameter space where $V_0(\varphi)$ and $V_L(\varphi', \xi)$ have the minima given in eq. (29). Notice that in the present setup dealing with a discrete symmetry such as A4 provides a great advantage as far as the alignment problem is concerned. A continuous flavour symmetry such as,

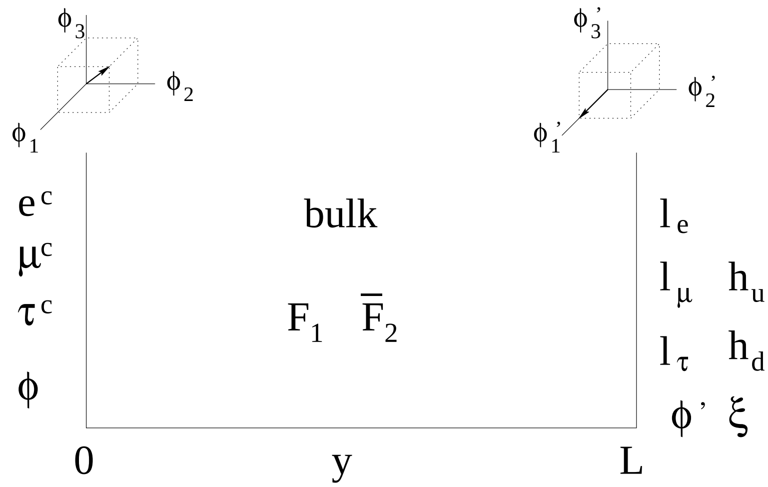

Figure 2: Fifth dimension and localization of scalar and fermion fields. The symmetry breaking sector includes the A4 triplets φ and φ', localized at the opposite ends of the interval. Their VEVs are dynamically aligned along the directions shown at the top of the figure.

for instance, SO(3) would need some extra structure to achieve the desired alignment. Indeed the potential energy $\int d^4x[V_0(\varphi)+V_L(\varphi',\xi)]$ would be invariant under a much bigger symmetry, $SO(3)_0\times$ $SO(3)_L$, with the $SO(3)_0$ acting on φ and leaving (φ',ξ) invariant and vice-versa for $SO(3)_L$. This symmetry would remove any alignment between the VEVs of φ and those of (φ',ξ). If, for instance, (29) is minimum of the potential energy, then any other configuration obtained by acting on (29) with $SO(3)_0\times$ $SO(3)_L$ would also be a minimum and the relative orientation between the two sets of VEVs would be completely undetermined. A discrete symmetry such as A4 has not this problem, because applying separate A4 transformation on the minimum solutions on each brane a finite number of degenerate vacua is obtained which can be shown to correspond to the same physics apart from redefinitions of fields and parameters.

6. A4 model with SUSY in 4 Dimensions

We now discuss an alternative supersymmetric solution to the vacuum alignment problem [18)]. In a SUSY context, the right-hand side of eq. (28) should be interpreted as the superpotential w_l of the theory, in the lepton sector:

$$\begin{aligned} w_l &= y_e e^c(\varphi l) + y_\mu \mu^c(\varphi l)" + y_\tau \tau^c(\varphi l)' + \\ &+ (x_a\xi + \tilde{x}_a\tilde{\xi})(ll) + x_b(\varphi' ll) + h.c. + ... \end{aligned} \tag{44}$$

where dots stand for higher dimensional operators and where we have also added an additional A4-invariant singlet $\tilde{\xi}$. Such a singlet does not modify the structure of

the mass matrices discussed previously, but plays an important role in the vacuum alignment mechanism. A key observation is that the superpotential w_l is invariant not only with respect to the gauge symmetry SU(2)× U(1) and the flavour symmetry $\mathrm{U}(1)_F \times A_4$, but also under a discrete Z_3 symmetry and a continuous $\mathrm{U}(1)_R$ symmetry under which the fields transform as shown in the following table.

Field	l	e^c	μ^c	τ^c	$h_{u,d}$	φ	φ'	ξ	$\tilde{\xi}$	φ_0	φ_0'	ξ_0
A4	3	1	$1'$	$1''$	1	3	3	1	1	3	3	1
Z_3	ω	ω^2	ω^2	ω^2	1	1	ω	ω	ω	1	ω	ω
$U(1)_R$	1	1	1	1	0	0	0	0	0	2	2	2

We see that the Z_3 symmetry explains the absence of the term (ll) in w_l: such a term transforms as ω^2 under Z_3 and need to be compensated by the field ξ in our construction. At the same time Z_3 does not allow the interchange between φ and φ', which transform differently under Z_3. The singlets ξ and $\tilde{\xi}$ have the same transformation properties under all symmetries and, as we shall see, in a finite range of parameters, the VEV of $\tilde{\xi}$ vanishes and does not contribute to neutrino masses. Charged leptons and neutrinos acquire masses from two independent sets of fields. If the two sets of fields develop VEVs according to the alignment described in eq. (29), then the desired mass matrices follow.

Finally, there is a continuous $U(1)_R$ symmetry that contains the usual R-parity as a subgroup. Suitably extended to the quark sector, this symmetry forbids the unwanted dimension two and three terms in the superpotential that violate baryon and lepton number at the renormalizable level. The $U(1)_R$ symmetry allows us to classify fields into three sectors. There are "matter fields" such as the leptons l, e^c, μ^c and τ^c, which occur in the superpotential through bilinear combinations. There is a "symmetry breaking sector" including the higgs doublets $h_{u,d}$ and the flavons φ, φ', $(\xi, \tilde{\xi})$. Finally, there are "driving fields" such as φ_0, φ_0' and ξ_0 that allows to build a non-trivial scalar potential in the symmetry breaking sector. Since driving fields have R-charge equal to two, the superpotential is linear in these fields.

The full superpotential of the model is

$$w = w_l + w_d \tag{45}$$

where, at leading order in a $1/\Lambda$ expansion, w_l is given by eq. (44) and the "driving" term w_d reads:

$$\begin{aligned} w_d &= M(\varphi_0\varphi) + g(\varphi_0\varphi\varphi) + g_1(\varphi_0'\varphi'\varphi') + g_2\tilde{\xi}(\varphi_0'\varphi') + g_3\xi_0(\varphi'\varphi') \\ &+ g_4\xi_0\xi^2 + g_5\xi_0\xi\tilde{\xi} + g_6\xi_0\tilde{\xi}^2 \quad . \end{aligned} \tag{46}$$

At this level there is no fundamental distinction between the singlets ξ and $\tilde{\xi}$. Thus we are free to define $\tilde{\xi}$ as the combination that couples to $(\varphi_0'\varphi')$ in the superpotential w_d. We notice that at the leading order there are no terms involving the Higgs fields

$h_{u,d}$. We assume that the electroweak symmetry is broken by some mechanism, such as radiative effects when SUSY is broken. It is interesting that at the leading order the electroweak scale does not mix with the potentially large scales u, v and v'. The scalar potential is given by:

$$V = \sum_i \left|\frac{\partial w}{\partial \phi_i}\right|^2 + m_i^2 |\phi_i|^2 + ... \tag{47}$$

where ϕ_i denote collectively all the scalar fields of the theory, m_i^2 are soft masses and dots stand for D-terms for the fields charged under the gauge group and possible additional soft breaking terms. Since m_i are expected to be much smaller than the mass scales involved in w_d, it makes sense to minimize V in the supersymmetric limit and to account for soft breaking effects subsequently. A detailed minimization analysis, presented in ref.[18)], shows the the desired alignment solution is indeed realized. In ref.[19)] we have shown that it is straightforward to reformulate this SUSY model in the approach where the A4 symmetry is derived from orbifolding.

7. Corrections to the Lowest Approximation

The results of the previous sections hold to first approximation. Higher-dimensional operators, suppressed by additional powers of the cut-off Λ, can be added to the leading terms in the lagrangian. These corrections have been classified and discussed in detail in refs.[17)], [18)]. They are completely under control in our models and can be made negligibly small without any fine-tuning: one only needs to assume that the VEV's are sufficiently smaller than the cutoff Λ. Higher-order operators contribute corrections to the charged lepton masses, to the neutrino mass matrix and to the vacuum alignment. These corrections, suppressed by powers of VEVs/Λ, with different exponents in different versions of A4 models, affect all the relevant observable with terms of the same order: s_{13}, s_{12}, s_{23}, r. If we require that the subleading terms do not spoil the leading order picture, these deviations should not be larger than about 0.05. This can be inferred by the agreement of the HPS value of $\tan^2\theta_{12}$ with the experimental value, from the present bound on θ_{13} or from requiring that the corrections do not exceed the measured value of r. In the SUSY model, where the largest corrections are linear in VEVs/Λ [18)], this implies the bound

$$\frac{v_S}{\Lambda} \approx \frac{v_T}{\Lambda} \approx \frac{u}{\Lambda} < 0.05 \tag{48}$$

which does not look unreasonable, for example if VEVs$\sim M_{GUT}$ and $\Lambda \sim M_{Planck}$.

8. See-saw Realization

We can easily modify the previous model to implement the see-saw mechanism [18)]. We introduce conjugate right-handed neutrino fields ν^c transforming as a triplet

of A4 and we modify the transformation law of the other fields according to the following table:

Field	ν^c	φ'	ξ	$\tilde{\xi}$	φ'_0	ξ_0
A4	3	3	1	1	3	1
Z_3	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2
$U(1)_R$	1	0	0	0	2	2

The superpotential becomes

$$w = w_l + w_d \tag{49}$$

where the 'driving' part is unchanged, whereas w_l is now given by:

$$\begin{aligned} w_l &= y_e e^c(\varphi l) + y_\mu \mu^c(\varphi l)" + y_\tau \tau^c(\varphi l)' + y(\nu^c l) + (x_A \xi + \tilde{x}_A \tilde{\xi})(\nu^c \nu^c) \\ &+ x_B(\varphi' \nu^c \nu^c) + h.c. + ... \end{aligned} \tag{50}$$

dots denoting higher-order contributions. The vacuum alignment proceeds exactly as discussed in section 8 and also the charged lepton sector is unaffected by the modifications. In the neutrino sector, after electroweak and A4 symmetry breaking we have Dirac and Majorana masses:

$$m_\nu^D = y v_u \mathbf{1}, \quad M = \begin{pmatrix} A & 0 & 0 \\ 0 & A & B \\ 0 & B & A \end{pmatrix} u \quad , \tag{51}$$

where $\mathbf{1}$ is the unit 3×3 matrix and

$$A \equiv 2x_A \quad , \qquad B \equiv 2x_B \frac{v_S}{u} \quad . \tag{52}$$

The mass matrix for light neutrinos is $m_\nu = (m_\nu^D)^T M^{-1} m_\nu^D$ with eigenvalues

$$m_1 = \frac{y^2}{A+B}\frac{v_u^2}{u} \quad , \qquad m_2 = \frac{y^2}{A}\frac{v_u^2}{u} \quad , \qquad m_3 = \frac{y^2}{-A+B}\frac{v_u^2}{u} \quad . \tag{53}$$

The mixing matrix is the HPS one, eq. (3). In the presence of a see-saw mechanism both normal and inverted hierarchies in the neutrino mass spectrum can be realized. If we call Φ the relative phase between the complex number A and B, then $\cos\Phi > -|B|/2|A|$ is required to have $|m_2| > |m_1|$. In the interval $-|B|/2|A| < \cos\Phi \leq 0$, the spectrum is of inverted hierarchy type, whereas in $|B|/2|A| \leq \cos\Phi \leq 1$ the neutrino hierachy is of normal type. It is interesting that this model is an example of model with inverse hierarchy, realistic θ_{12} and θ_{23} and, at least in a first approximation, $\theta_{13} = 0$. The quantity $|B|/2|A|$ cannot be too large, otherwise the ratio r cannot be reproduced. When $|B| \ll |A|$ the spectrum is quasi degenerate. When $|B| \approx |A|$ we

obtain the strongest hierarchy. For instance, if $B = -2A + z$ ($|z| \ll |A|, |B|$), we find the following spectrum:

$$
\begin{aligned}
|m_1|^2 &\approx \Delta m^2_{atm}(\frac{9}{8} + \frac{1}{12}r), \\
|m_2|^2 &\approx \Delta m^2_{atm}(\frac{9}{8} + \frac{13}{12}r), \\
|m_3|^2 &\approx \Delta m^2_{atm}(\frac{1}{8} + \frac{1}{12}r).
\end{aligned}
\tag{54}
$$

When $B = A + z$ ($|z| \ll |A|, |B|$), we obtain:

$$
\begin{aligned}
|m_1|^2 &\approx \Delta m^2_{atm}(\frac{1}{3}r), \\
|m_2|^2 &\approx \Delta m^2_{atm}(\frac{4}{3}r), \\
|m_3|^2 &\approx \Delta m^2_{atm}(1 - \frac{1}{3}r).
\end{aligned}
\tag{55}
$$

These results are affected by higher-order corrections induced by non renormalizable operators with similar results as in the version with no see-saw. In conclusion, the symmetry structure of the model is fully compatible with the see-saw mechanism.

9. Quarks

To include quarks the simplest possibility is to adopt for quarks the same classification scheme under A4 that we have used for leptons. Thus we tentatively assume that left-handed quark doublets q transform as a triplet 3, while the right-handed quarks (u^c, d^c), (c^c, s^c) and (t^c, b^c) transform as 1, 1$'$ and 1", respectively. We can similarly extend to quarks the transformations of Z_3 and $\mathrm{U}(1)_R$ given for leptons in the table of section 6. Such a classification for quarks leads to a diagonal CKM mixing matrix in first approximation [14,15,18]. In fact, proceeding as described in detail for the lepton sector, one immediately obtains that the up quark and down quark mass matrices are made diagonal by the same unitary transformation given in eq.(33). Thus $U_u = U_d$ and $V_{CKM} = U_u^\dagger U_d = 1$ in leading order, providing a good first order approximation. Like for charged leptons, the quark mass eigenvalues are left unspecified by A4 and their hierarchies can be accomodated by a suitable $\mathrm{U}(1)_F$ set of charge assignments for quarks.

The problems come when we discuss non-leading corrections. As seen in section 7, first-order corrections to the lepton sector should be typically below 0.05, approximately the square of the Cabibbo angle. Also, by inspecting these corrections more closely, we see that, up to very small terms [18], all corrections are the same in the up and down sectors and therefore they almost exactly cancel in the mixing matrix V_{CKM}. We conclude that, if one insists in adopting for quarks the same flavour prop-

erties as for leptons, than new sources of A4 breaking are needed in order to produce large enough deviations of V_{CKM} from the identity matrix.

The A4 classification for quarks and leptons discussed in this section, which leads to an appealing first approximation with $V_{CKM} \sim 1$ for quark mixing and to U_{HPS} for neutrino mixings, is not compatible with A4 commuting with SU(5) or SO(10). In fact for this to be true all particles in a representation of SU(5) should have the same A4 classification. But, for example, both the $Q = (u, d)_L$ LH quark doublet and the RH charged leptons l^c belong to the 10 of SU(5), yet they have different A4 transformation properties. Note that the A4 classification is instead compatible with the Pati-Salam group SU(4)xSU(2)xSU(2) [24)]

Recent directions of research include the study of different finite groups for tribimaximal mixing, generally larger than A4 [25)], the attempt of improving the quark mixing description while keeping the good features of A4 [26,27)] and the construction of GUT models with approximate tribimaximal mixing [29)].

In ref.[26,27)] the double covering group of A4, called T' (or also $SL_2(F_3)$), was considered to construct a model which is identical to A4 in the lepton sector while it is better in the quark sector. Here we follow ref.[27)]. The group T' has 24 transformations and its irreducible, inequivalent representations are 1, 1', 1", 2, 2', 2", 3. While A4 is not a subgroup of T', the latter group can reproduce all the good results of A4 in the lepton sector, where one restricts to the singlet and triplet representations. For quarks one can use singlet and doublet representations. Precisely, the quark doublet and the antiquarks of the 3rd generations are each classified in 1, while the other quark doublets and the antiquarks are each in a 2" that includes the 1st and 2nd generations. The separation of the 3 families in a 1+2 of U(2) was already considered in ref.[28)]. An advantage of this classification of top and bottom quarks as singlets is that they acquire mass already at the renormalisable vertex level, thus providing a rationale for their large mass. Moreover the model, through additional parity symmetries, is arranged in such a way that the flavons that break A4 in the neutrino sector do not couple to quarks in leading order, while the triplet flavon that enters the mass matrix of charged leptons couples to two quark 2" doublets to give an invariant mass term that leads to c and s quark masses. An additional doublet flavon which has no effect in the lepton sector, introduces by its vev the mixing between the 2nd and 3rd family. Finally masses and mixings for the 1st generation are due to subleading effect.

The T' model provides a combination of the lepton sector as successfully described in A4 with a reasonable description of the quark sector (where some amount of fine tuning is however still needed). But the classification of quarks and leptons of the T' model is again not compatible with a direct embedding in GUT's because it does not commute with SU(5). The problem of a satisfactory Grand Unified version of

tribimaximal mixing is still open. Attempts in this direction are given in refs.[29].

10. Conclusion

In the last decade we have learnt a lot about neutrino masses and mixings. A list of important conclusions have been reached. Neutrinos are not all massless but their masses are very small. Probably masses are small because neutrinos are Majorana particles with masses inversely proportional to the large scale M of lepton number violation. It is quite remarkable that M is empirically close to $10^{14-15} GeV$ not far from M_{GUT}, so that neutrino masses fit well in the SUSY GUT picture. Also out of equilibrium decays with CP and L violation of heavy RH neutrinos can produce a B-L asymmetry, then converted near the weak scale by instantons into an amount of B asymmetry compatible with observations (baryogenesis via leptogenesis) [30]. It has been established that neutrinos are not a significant component of dark matter in the Universe. We have also understood there there is no contradiction between large neutrino mixings and small quark mixings, even in the context of GUTs.

This is a very impressive list of achievements. Coming to a detailed analysis of neutrino masses and mixings a very long collection of models have been formulated over the years. With continuous improvement of the data and more precise values of the mixing angles most of the models have been discarded by experiment. Still the missing elements in the picture like, for example, the scale of the average neutrino m^2, the pattern of the spectrum (degenerate or inverse or normal hierarchy) and the value of θ_{13} have left many different viable alternatives for models. It certainly is a reason of satisfaction that so much has been learnt recently from experiments on neutrino mixings. By now, besides the detailed knowledge of the entries of the V_{CKM} matrix we also have a reasonable determination of the neutrino mixing matrix U_{P-MNS}. It is remarkable that neutrino and quark mixings have such a different qualitative pattern. One could have imagined that neutrinos would bring a decisive boost towards the formulation of a comprehensive understanding of fermion masses and mixings. In reality it is frustrating that no real illumination was sparked on the problem of flavour. We can reproduce in many different ways the observations but we have not yet been able to single out a unique and convincing baseline for the understanding of fermion masses and mixings. In spite of many interesting ideas and the formulation of many elegant models, some of them reviewed here, the mysteries of the flavour structure of the three generations of fermions have not been much unveiled.

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